7.3 The first 5 of the 10 Rules of Replacement

With these rules, the premise is logically equivalent to the conclusion. So you can **replace** one with the other no matter where it is. That means you can do these last 10 rules to parts of larger statements leaving the rest the same. (Only the last 10 rules are Rules of replacement, not the first 8). The 4 little dots means “logically equivalent”. On your rule sheet from the Course Packet, they are replaced by the equal sign (=).

**DeMorgan’s (DM)**

\[
\neg(p \lor q) \equiv \neg p \land \neg q \\
\neg(p \land q) \equiv \neg p \lor \neg q
\]

We saw this rule in action when we were doing translations. I.e. “Neither A nor B ate the pizza” \(\neg(A \lor B)\) is the same thing as saying that “A didn’t eat the pizza AND B didn’t eat the pizza.” \(\neg A \land \neg B\)

DM examples:
a. \(\neg(O \land \neg H) \equiv \neg O \lor H\) (or \(\neg O \lor \neg \neg H\))
b. \(E \land G \equiv \neg(E \land \neg G)\)
c. \(\neg(H \land N) \land Y \equiv (H \land \neg N) \land Y\) (Notice that we only DM-ed the “p” and left the main connective dot the same. We could have DM-ed the main connective dot only like in example d.)
d. \(\neg(H \land N) \land Y \equiv \neg[(H \land N) \land \neg Y]\) (the tilde on the outside of the brackets is the main connective.)
e. \(W \supset (\neg T \lor K) \equiv W \supset (T \land \neg K)\) (notice that the “\(\supset\)” stayed the same)

**If you start with a dot or wedge statement, you will end up with a tilde on the outside of parentheses.** In “d” above, after the DM you would NOT be able to simp. b/c the dot is not the main connective anymore.

**One way to do DM is to replace everything there with its opposite.** There are 3 things to replace:

1. First put down the opposite of the whole statement (take a tilde off the outside of parentheses, OR put one on the outside of parentheses.)
2. Then replace the “p” with its opposite (add or take off a tilde).
3. Then replace the “q” with its opposite (add or take off a tilde).

Finally change the dot to wedge or vice versa.
The other way is to factor the tilde through if you start with a \( \sim(\quad) \) statement, or “unfactor” it if you start with a wedge or dot statement. If you do it this way, it may help to Double Negate the positive letter first:

Take the example from above. To double negate each positive statement you would get \( \sim\sim E \cdot \sim\sim G \) then take a tilde off each and move it to the front of parentheses, leave one tilde on each of the letters, and change the sign from dot to wedge: \( \sim(\sim E \vee \sim G) \)

**DM is a very important rule.** Notice in your ch.7 studyguide, it is #4 of the five rules (out of 18) that I talk about in detail. You will be DM-ing a lot. Make sure you do it properly. See the DM worksheet in Course packet. See also p.16 in the course packet for an example of every possible way to do DM. See also the pdf on the website about doing DM properly.

**Commutativity (Com)** This rule allows you to switch around any dot or wedge no matter where it is. This works ONLY for dots and wedges. NEVER do Com to a horseshoe!!!!!

\[
\begin{align*}
p \vee q & \equiv q \vee p \\
p \cdot q & \equiv q \cdot p
\end{align*}
\]

Com examples:

1. \((T \cdot G) \vee \sim B\) /concl.
2. \((G \cdot T) \vee \sim B\) 1 com (switching around dot)
3. \(\sim B \vee (T \cdot G)\) 1 com (switching around wedge)
4. \(\sim B \vee (G \cdot T)\) 1 com (switching around both)

You can com line 1 any of these 3 ways.

**Associativity (Assoc):** This rule lets you move the parentheses from right to left and vice versa as long as the statement is ALL wedges or ALL dots. There is no switching involved here. The only thing that changes is the parentheses.

\[
\begin{align*}
p \vee (q \vee r) & \equiv (p \vee q) \vee r \\
p \cdot (q \cdot r) & \equiv (p \cdot q) \cdot r
\end{align*}
\]

EX.)

\[
\begin{align*}
\sim W \vee (\sim R \vee H) & \equiv (\sim W \vee \sim R) \vee H \\
E \cdot (F \cdot \sim A) & \equiv (E \cdot F) \cdot \sim A
\end{align*}
\]
Note: You CANNOT do Assoc. to a statement like this one:  Y v ~(I v H) ---because of the tilde on the outside of the parentheses. **If there is a tilde on the outside of parentheses, you can NEVER move the parentheses.**

**Distribution (Dist)** is for statements with a mix of dots AND wedges.

\[
\begin{align*}
p v (q \cdot r) & \equiv (p v q) \cdot (p v r) \\
p \cdot (q v r) & \equiv (p \cdot q) v (p \cdot r)
\end{align*}
\]

Essentially you are distributing the “p” through the parentheses. The wedge and dot switch out. i.e. If the wedge was the main connective, it will go inside the parentheses and the dot which was inside parentheses will come out and become the new main connective.

i.e.  \( Y v (~E \cdot H) \equiv (Y v ~E) \cdot (Y v H) \)

i.e.  \( ~M \cdot (O v ~Z) \equiv (~M \cdot O) v (~M \cdot ~Z) \)

**Again, you cannot do this rule with a tilde on the outside of your parentheses.**

**Doing a “backwards Dist” (from right to left).**

A strategy you’ll want to know is that if you ever have 2 wedge statements that share a letter in common (not opposites, but the same letter), then you’ll be able to conjoin them and do a backwards Dist. I.E. In the following proof your two wedge statements have the ‘U” in common:

1.  \( U v ~R \)
2.  \( ~H v U \quad / \quad U v ~(R v H) \quad \rightarrow \) This is the sort of statement you’ll see a lot.
3.  \( U v ~H \quad \) 2 com.
4.  \( (U v ~R) \cdot (U v ~H) \quad \) 1,3 conj.
5.  \( U v (~R \cdot ~H) \quad \) 4 dist.
6.  \( U v ~(R v H) \quad \) 5 DM

**Double Negation (DN):** This rule is optional. If you like your statements to fit the actual rules as written in textbook, then you may want to use this rule.

\[ p \equiv \neg \neg p \]
Note: **DN means you either take TWO tildes off, or you put TWO tildes on.** So if you do DN to ~R you get ~~~R. **You can NEVER get the opposite of something using DN!!**

The tilde’s have to be right next to each other to DN. You cannot have them separated by a parentheses.

i.e. To make your MT look more like the rule (where the tilde is on the second line):

1. ~H ⊃ ~K                    (If Horace can’t come over then Katy also cannot come over)
2. K                           (Katy can come over)
3. ~~K  2 DN                  (which means the same as: It’s NOT the case that Katy CANNOT come over)
4. ~~H  1,3 MT                (So with MT, it’s NOT the case that Horace CANNOT come over)
5. H  4 DN                    (Which means that Horace CAN come over)

**Strategy F/12:** This is a particular strategy used often in the 7.3 proofs. It is “F” in my course packet, but “12” in the book.

Here’s what the format looks like:

<table>
<thead>
<tr>
<th>Step</th>
<th>Statement</th>
<th>Justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>~p</td>
<td></td>
</tr>
<tr>
<td>2.</td>
<td>~p ∨ ~q</td>
<td>1 Add</td>
</tr>
<tr>
<td>3.</td>
<td>~(p ∨ q)</td>
<td>2 DM</td>
</tr>
<tr>
<td>4.</td>
<td>~(p ∨ q)</td>
<td>3 DM</td>
</tr>
</tbody>
</table>

If you need the negation of a p ∨ q AND you have the opposite of either the p or q, then you need to ADD the OPPOSITE of the other one, and DM it.

See proofs that use this strategy below.

Examples of proofs:

Note: Start using scrap paper. Break down the conclusion as far as you can on the scrap paper if it is a fairly complicated statement (i.e. Do dist or DM etc to it). Figure out what you want by looking at the premises, then figure out what statement you NEED in order to get what you want (i.e If it’s in the right side of a horseshoe statement, you NEED the matching left side to do MP. If it’s in the left side of a horseshoe statement, you NEED the opposite of the right side to get the opposite of the left. If it’s in either side of a wedge statement, you NEED the opposite of the other side.) Write this statement down on the scrap paper! Do stuff to it (like DM etc.). Unless you can do the proof in your head, using scrap paper helps you by giving you more info to look at.

For the proof below, I would first break down the conclusion by doing Dist. to it. That shows me that what I need are the “q’s” to lines 2 and 3 (MP is the only way to get those so you need the matching “p’s”). Line 1 looks like a good DM (you should always DM any negated wedge statement even if you just do it on your scrap paper.)
A.  1. \(~(\neg H \lor Z)\)
2. \((H \lor M) \supset (T \lor R)\)
3. \((O \lor \neg Z) \supset (T \lor \neg S)\)
   \hspace{1cm} / \hspace{1cm} \neg T = (T \lor R) \cdot (T \lor S) \hspace{1cm} \text{(dist)}
4. \(H \cdot \neg Z\)
   \hspace{1cm} 1 \text{ DM}
5. \(H\)
   \hspace{1cm} 4 \text{ simp.}
6. \(\neg Z\)
   \hspace{1cm} 4 \text{ simp.}
7. \(H \lor M\)
   \hspace{1cm} 5 \text{ Add}
8. \(T \lor R\)
   \hspace{1cm} 2,7 \text{ MP (Here’s half of your concl.)}
9. \(\neg Z \lor O\)
   \hspace{1cm} 6 \text{ Add}
10. \(O \lor \neg Z\)
    \hspace{1cm} 10 \text{ Com}
11. \(T \lor \neg S\)
    \hspace{1cm} 3,10 \text{ MP (Here’s the other half of your concl.)}
12. \((T \lor R) \cdot (T \lor S)\)
    \hspace{1cm} 8,11 \text{ conj.}
13. \(T \lor (R \cdot S)\)
    \hspace{1cm} 12 \text{ Dist.}

The following 3 proofs require Strategy F from the Course Packet (in red). Str. F basically shows how you can get \((\neg p \cdot q)\) from either \(\neg p\) or \(\neg q\) by adding the opposite of the other one and doing DM. If you want a \((\neg p \lor q)\), you’ll need to have both \(\neg p\) and \(\neg q\) to start with.

B.  1. \((B \lor \neg Z) \supset (R \cdot T)\)
2. \(\neg R\)
   \hspace{1cm} / \hspace{1cm} \neg (B \lor R) = (\neg B \cdot \neg R) \hspace{1cm} \text{(DM)}
3. \(\neg R \lor \neg T\)
   \hspace{1cm} 2 \text{ Add}
4. \(\neg (R \cdot T)\)
   \hspace{1cm} 3 \text{ DM}
5. \(\neg (B \lor \neg Z)\)
   \hspace{1cm} 1,4 \text{ MT}
6. \(\neg B \cdot \neg Z\)
   \hspace{1cm} 5 \text{ DM}
7. \(\neg B\)
   \hspace{1cm} 6 \text{ simp.}
8. \(\neg B \cdot \neg R\)
   \hspace{1cm} 2,7 \text{ conj.}
9. \(\neg (B \lor R)\)
   \hspace{1cm} 8 \text{ DM}

Notice that I DM-ed the conclusion on the side (scrap paper). That tells me I need to find a \(\neg B\) and a \(\neg R\) by themselves first.

I already have the \(\neg R\) on line 2, so all I need is the \(\neg B\). \(\neg B\) is in the left half of a \(p \supset q\) statement (line 1). **If I want anything from the left half of a horseshoe, I need to first have the OPPOSITE of the right half of the horseshoe.** In this case, I need \(\neg (R \cdot T)\) to do MT with line 1. I DM \(\neg (R \cdot T)\) and get \(\neg R \lor \neg T\). Since I have the \(\neg R\) on line 2, I can add the \(\neg T\) to it, and DM it. Then I can do the MT and get the OPPOSITE of “p” or \(\neg (B \lor \neg Z)\) by itself. DM-ing that will give me a dot I can simp my \(\neg B\) from.

C.  1. \((Q \lor \neg Z) \lor (\neg O \cdot T)\)
2. \(Q \lor \neg A\)
3. \(\neg T\)
   \hspace{1cm} / \hspace{1cm} \neg (A \lor Z) = Q \lor (\neg A \cdot \neg Z) = (Q \lor \neg A) \cdot (Q \lor \neg Z)
4. \(\neg T \lor O\)
   \hspace{1cm} 3 \text{ Add}
5. \(O \lor \neg T\)
   \hspace{1cm} 4 \text{ com}
6. \(\neg (\neg O \cdot T)\)
   \hspace{1cm} 5 \text{ DM}
7. \(Q \lor \neg Z\)
   \hspace{1cm} 1,6 \text{ DS}
8. \((Q \lor \neg A) \cdot (Q \lor \neg Z)\)
   \hspace{1cm} 2,7 \text{ conj}
9. \(Q \lor (\neg A \cdot \neg Z)\)
   \hspace{1cm} 8 \text{ Dist.}
10. \(Q \lor (A \lor Z)\)
    \hspace{1cm} 9 \text{ DM}

Notice that I first DM-ed the right half of the conclusion, then I did Distribution to it. That shows me that I need to get \((Q \lor \neg A)\) and \((Q \lor \neg Z)\) by themselves. I see that I already have the \(Q \lor \neg A\) on line 2, so all I need to find is the \(Q \lor \neg Z\). It is the “p” of a \(p \lor q\) statement on line 1. To get the side I want by itself, I need the OPPOSITE of the other side which is \(\neg (\neg O \cdot T)\). That DM-ed is \(O \lor \neg T\). I have \(\neg T\) on line 3, so I can add the \(\neg O\) to it, com, and DM. Then I can do the DS with line 1.
D. 1. \((B \lor \neg Z) \lor O\)
2. \((T \land \neg B) \supset \neg (Z \lor O)\)
3. \neg B / \neg (T \land M) = \neg T \lor \neg M (DM)
4. \(B \lor (\neg Z \lor O)\) 1 assoc.
5. \(Z \lor O\) 3, 4 DS
6. \(\neg (T \land \neg B)\) 2, 5 MT
7. \(\neg T \land B\) 6 DM
8. \(\neg T\) 3, 7 DS
9. \(\neg T \land \neg M\) 8 add
10. \(\neg (T \land M)\) 9 DM

E. 1. \neg [K \cdot (O \lor M)]
2. \((A \cdot K) \supset (K \cdot O)\) / \neg [K \cdot (A \lor M)]
3. \neg K \land (O \land M) 1 DM
4. \neg K \land (O \land M) 3 DM
5. \((\neg K \land \neg O) \land (\neg K \land \neg M)\) 4 Dist.
6. \neg K \land \neg O 5 simp.
7. \neg K \land \neg M 5 simp.
8. \neg (K \land O) 6 DM
9. \neg (A \land K) 2, 8 MT
10. \neg A \land \neg K 9 DM
11. \neg K \land \neg A 10 com
12. \((\neg K \land \neg A) \land (\neg K \land \neg M)\) 7, 11 conj.
13. \neg K \land (\neg A \land \neg M) 12 Dist.
14. \neg K \land (A \land M) 13 DM
15. \neg [K \cdot (A \land M)] 14 DM

With a conclusion like this, you should definitely break it down. Do DM to it to get a wedge. ~K \land \neg (A \lor M)
Then DM again to the right side to set up for a Dist. ~K \land \neg (A \land \neg M)
Then Dist to get a dot you can simp. both sides of. (~K \land \neg A) \land (\neg K \land \neg M)
These two statements are what you are looking for in your premises.

Make sure you do the DM worksheet in the course packet and download the answer key from the website.